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A new formulation of the field equations for the stationary axisymmetric vacuum gravitational field I. General theory

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Abstract. A new formulation of the stationary axisymmetric vacuum gravitational field equations which is substantially different from the well known formulations of Lewis and Ernst is presented. The basic variable is $e^{2\gamma}=-g_{11}g_{44}$ and satisfies a field equation of the fourth differential order which may be interpreted as the condition that a certain 2-space has constant curvature, $K=-1$. The principal motivation is that for many known solutions and all known asymptotically flat (non-static) solutions, $e^{2\gamma}$ takes a much simpler functional form than either the metric coefficients, g_{44}, g_{34} and g_{33}, or the Ernst potentials, $\mathscr{E}$ and ξ. Three methods are given for the construction of the full metric from $e^{2\gamma}$. A duality principle is invoked to provide a very similar field equation for the metric coefficient, $e^{2\gamma-2u}=-g_{11}$.

1. Introduction and preliminary

In this paper, we show how to construct solutions of Einstein's equations for the stationary axisymmetric vacuum gravitational field from solutions of the fourth-order quasi-linear partial differential equation,

$$\begin{aligned}J(2A_{zz}+2C_{rr}-4B_{rz})-J_zA_z-J_rC_r-4J^2&\\ -BA_rC_z+BA_zC_r+2CA_rB_z+2AC_zB_r-4BB_rB_z&=0,\end{aligned}\tag{1.1}$$

where

$$A=-2(\gamma_{rr}+\gamma_{zz})+(2/r)\gamma_r,\qquad B=(2/r)\gamma_z,$$

$$C=-2(\gamma_{rr}+\gamma_{zz})-(2/r)\gamma_r,\qquad J=AC-B^2.$$

(Subscripts denote partial differentiation, e.g. $\gamma_r\equiv\partial\gamma/\partial r$, $\gamma_{rr}\equiv\partial^2\gamma/\partial r^2$.) This function γ appears in the metric coefficients, g_{11} and $g_{22}=g_{11}$, in the Weyl–Lewis–Papapetrou canonical form,

$$ds^2=e^{2u}(dt-\omega\,d\phi)^2-e^{-2u}\{e^{2\gamma}(dr^2+dz^2)+r^2\,d\phi^2\},\tag{1.2}$$

where $(x^1,x^2,x^3,x^4)=(r,z,\phi,t)$ are cylindrical co-ordinates and time and u, ω and γ are functions of r and z only. The same equation (1.1) is also satisfied by $\gamma-u+\frac{1}{4}\ln r$.

Just as Ernst's (1968) complex potential formulation has enabled the discovery of many interesting new solutions, this present formulation should assist in constructing relatively complicated new solutions from simple assumptions. One assumption which we shall explore in the accompanying paper (Cosgrove 1978a, to be referred to as II) is the possibility that $e^{2\gamma}$ be a function of one variable only or depend in a simple manner on a second variable so that (1.1) reduces to an ordinary differential equation of the fourth order. The six-parameter generalised Tomimatsu–Sato solutions (Cosgrove 1977a, 1978b) and contractions such as the rotating Curzon metric (Cosgrove 1977b), a recently published family of Ernst (1977) and a new family in § 3.3 of II are all of this form. In fact, it is clear from the tabulated forms of $e^{2\gamma}$ in the original paper of Tomimatsu and Sato (1973) and is a direct consequence of their 'Rule (a)' that $e^{2\gamma}$ is a function of the single variable $\eta = (x^2-1)/(1-y^2)$, where (x, y) are the usual spheroidal co-ordinates.

It will be necessary, first, to write down complete sets of field equations in both the metric tensor and Ernst potential formulations and discuss some of their transformation properties. A complete set of Einstein's vacuum field equations for the metric (1.2) is

$$u_{rr}+u_{zz}+(1/r)u_r+(1/2r^2)\,e^{4u}(\omega_r^2+\omega_z^2)=0, \tag{1.3a}$$

$$\omega_{rr}+\omega_{zz}-(1/r)\omega_r+4u_r\omega_r+4u_z\omega_z=0, \tag{1.3b}$$

$$\gamma_r=r(u_r^2-u_z^2)-(1/4r)\,e^{4u}(\omega_r^2-\omega_z^2), \tag{1.4a}$$

$$\gamma_z=2ru_ru_z-(1/2r)\,e^{4u}\omega_r\omega_z, \tag{1.4b}$$

$$\gamma_{rr}+\gamma_{zz}+(1/r)\gamma_r=-2u_z^2-(1/2r^2)\,e^{4u}\omega_r^2. \tag{1.4c}$$

(This is essentially the (f, l, m) formulation of Lewis (1932) who had $f=g_{44}=e^{2u}$, $l=-g_{33}=r^2\,e^{-2u}-\omega^2\,e^{2u}$, $m=-g_{34}=\omega\,e^{2u}$.) These field equations remain invariant under the $SL(2)$ transformation group $\mathbf{L}$, given by

$$e^{2u'}=\beta_1^2\,e^{2u}-2\beta_1\beta_2\omega\,e^{2u}+\beta_2^2(\omega^2\,e^{2u}-r^2\,e^{-2u}), \tag{1.5a}$$

$$\omega'\,e^{2u'}=-\beta_1\beta_3\,e^{2u}+(\beta_1\beta_4+\beta_2\beta_3)\omega\,e^{2u}-\beta_2\beta_4(\omega^2\,e^{2u}-r^2\,e^{-2u}), \tag{1.5b}$$

$$\omega'^2\,e^{2u'}-r^2\,e^{-2u'}=\beta_3^2\,e^{2u}-2\beta_3\beta_4\omega\,e^{2u}+\beta_4^2(\omega^2\,e^{2u}-r^2\,e^{-2u}), \tag{1.5c}$$

$$e^{2\gamma'-2u'}=e^{2\gamma-2u}, \tag{1.5d}$$

which is represented by the unimodular matrix,

$$\begin{pmatrix}\beta_1 & \beta_2\\ \beta_3 & \beta_4\end{pmatrix}, \qquad \beta_1\beta_4-\beta_2\beta_3=1.$$

If this transformation is applied to the Weyl static solutions, there results the family of solutions found by Lewis (1932) while seeking solutions for which $r\,e^{-2u}$ and ω were functionally related. This family contains the rotating cylindrically symmetric solutions. However, the group $\mathbf{L}$ is equivalent to a co-ordinate transformation,

$$t=\beta_1t'+\beta_3\phi', \qquad \phi=\beta_2t'+\beta_4\phi', \tag{1.6}$$

i.e. a permutation of Killing vectors, so that it does not generate essentially new solutions. Its importance to us below is that it generates an equivalence class of metrics with the same $e^{2\gamma-2u}$.

Ernst's complex potential formulation (Ernst 1968, 1974) uses either of two complex functions,

$$\mathscr{E}=e^{2u}+i\psi, \qquad \xi=(1+\mathscr{E})/(1-\mathscr{E}), \tag{1.7}$$

where ψ is defined by the compatible relations,

$$\psi_r=(1/r)\,e^{4u}\omega_z, \qquad \psi_z=-(1/r)\,e^{4u}\omega_r. \tag{1.8}$$

We shall require Ernst's equations separated into real and imaginary parts as follows:

$$u_{rr}+u_{zz}+(1/r)u_r+\tfrac{1}{2}\,\mathrm{e}^{-4u}(\psi_r^2+\psi_z^2)=0, \tag{1.9a}$$

$$\psi_{rr}+\psi_{zz}+(1/r)\psi_r-4u_r\psi_r-4u_z\psi_z=0, \tag{1.9b}$$

$$\gamma_r=r(u_r^2-u_z^2)+\tfrac{1}{4}r\,\mathrm{e}^{-4u}(\psi_r^2-\psi_z^2), \tag{1.10a}$$

$$\gamma_z=2ru_ru_z+\tfrac{1}{2}r\,\mathrm{e}^{-4u}\psi_r\psi_z, \tag{1.10b}$$

$$\gamma_{rr}+\gamma_{zz}+(1/r)\gamma_r=-2u_z^2-\tfrac{1}{2}\,\mathrm{e}^{-4u}\psi_z^2. \tag{1.10c}$$

These equations are invariant under the $SL(2)$ transformation group, **P**, defined by

$$\mathrm{e}^{-2u'}=\alpha_1^2\,\mathrm{e}^{-2u}-2\alpha_1\alpha_2\psi\,\mathrm{e}^{-2u}+\alpha_2^2(\psi^2\,\mathrm{e}^{-2u}+\mathrm{e}^{2u}), \tag{1.11a}$$

$$\psi'\,\mathrm{e}^{-2u'}=-\alpha_1\alpha_3\,\mathrm{e}^{-2u}+(\alpha_1\alpha_4+\alpha_2\alpha_3)\psi\,\mathrm{e}^{-2u}-\alpha_2\alpha_4(\psi^2\,\mathrm{e}^{-2u}+\mathrm{e}^{2u}), \tag{1.11b}$$

$$\psi'^2\,\mathrm{e}^{-2u'}+\mathrm{e}^{2u'}=\alpha_3^2\,\mathrm{e}^{-2u}-2\alpha_3\alpha_4\psi\,\mathrm{e}^{-2u}+\alpha_4^2(\psi^2\,\mathrm{e}^{-2u}+\mathrm{e}^{2u}), \tag{1.11c}$$

$$\mathrm{e}^{2\gamma'}=\mathrm{e}^{2\gamma}, \tag{1.11d}$$

and represented by the unimodular matrix,

$$\begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_2 & \alpha_4\end{pmatrix}, \qquad \alpha_1\alpha_2-\alpha_2\alpha_3=1.$$

This transformation converts the Weyl solutions into the Papapetrou–Ehlers (PE) solutions, derived by Papapetrou (1953) by considering solutions with $u_r\omega_r+u_z\omega_z=0$ and by Ehlers (1959) (see also Ehlers and Kundt 1962) by seeking solutions with e^{2u} and ψ functionally related. Of the three independent parameters, two are trivial gauge parameters and the third is the so called NUT parameter. The case of a pure NUT transformation, with NUT parameter λ, is given by

$$\alpha_1=\alpha_4=\cos\tfrac{1}{2}\lambda, \qquad \alpha_2=-\alpha_3=\sin\tfrac{1}{2}\lambda. \tag{1.12a}$$

In this case, formulae (1.11*a*, *b*, *c*) simplify to

$$\xi'=\mathrm{e}^{\mathrm{i}\lambda}\xi. \tag{1.12b}$$

The similarity of these two sets of field equations and transformation groups suggests a direct mapping from one system to the other. This mapping is the group element, $I=I^{-1}$, of a finite group of order two, defined by

$$\mathrm{e}^{2u'}=r\,\mathrm{e}^{-2u}, \qquad \omega'=\mathrm{i}\psi, \qquad \psi'=-\mathrm{i}\omega, \qquad \mathrm{e}^{2\gamma'}=r^{1/2}\,\mathrm{e}^{2\gamma-2u}. \tag{1.13}$$

(Neugebauer and Kramer 1969, Sackfield 1975, Catenacci and Diaz Alonso 1976.) We shall say that two solutions related by (1.13) are 'dual' to one another. Definitions and theorems formulated in the Ernst system have duals in the metric components system and vice versa. In fact the groups **P** and **L** are dual to each other. If P is an element of **P** and L and element of **L**, then

$$ILI^{-1}=P, \qquad IPI^{-1}=L$$

where $\beta_1=\alpha_1$, $\beta_2=-\mathrm{i}\alpha_2$, $\beta_3=\mathrm{i}\alpha_3$, $\beta_4=\alpha_4$. Thus I maps PE solutions into Lewis solutions and vice versa. Although (1.13) involves complex numbers, it is often easy to generate real-valued solutions by simply letting parameters in the original solutions be complex (e.g., in the Kerr metric, let the angular momentum parameter be pure imaginary). We shall be concerned here primarily with the duality property and our first duality theorem is that γ and $\gamma-u+\frac{1}{4}\ln r$ satisfy the same fourth-order field equation.

2. Derivation of the γ equation

The field equation (1.1) for γ results from the elimination of u and ψ from the three equations (1.10a, b, c). First, let us see to what extent equations (1.9a, b) are deducible from (1.10a, b, c). Differentiate (1.10a) and (1.10b) to obtain expressions for γ_r, γ_z, γ_{rr}, γ_{zz}, γ_{rz} and γ_{zr} in terms of u and ψ and substitute into the two equations, $\gamma_{rz}-\gamma_{zr}=0$ and (1.10c). There results

$$\gamma_{rz}-\gamma_{zr}=-2ru_zE_1-\tfrac{1}{2}r\,\mathrm{e}^{-4u}\psi_zE_2=0,$$

$$\gamma_{rr}+\gamma_{zz}+(1/r)\gamma_r+2u_z^2+\tfrac{1}{2}\,\mathrm{e}^{-4u}\psi_z^2=2ru_rE_1+\tfrac{1}{2}r\,\mathrm{e}^{-4u}\psi_rE_2=0,$$

where E_1 and E_2 are the left hand sides of (1.9a) and (1.9b), respectively. Thus (1.10a, b, c) imply (1.9a, b) whenever $u_r\psi_z-u_z\psi_r\neq 0$, i.e. whenever u and ψ are not functionally related. However, as we shall see below, equations (1.10a, b, c) alone permit solutions for u and ψ which do not satisfy (1.9a, b) when u and ψ are functionally related. For the present, take $u_r\psi_z-u_z\psi_r\neq 0$.

To eliminate u and ψ, define

$$A=A_{[r,z]}=4u_r^2+\mathrm{e}^{-4u}\psi_r^2, \tag{2.1a}$$

$$B=B_{[r,z]}=4u_ru_z+\mathrm{e}^{-4u}\psi_r\psi_z, \tag{2.1b}$$

$$C=C_{[r,z]}=4u_z^2+\mathrm{e}^{-4u}\psi_z^2, \tag{2.1c}$$

$$D=D_{[r,z]}=\mathrm{e}^{-2u}(u_r\psi_z-u_z\psi_r), \tag{2.1d}$$

$$J=J_{[r,z]}=4D^2=AC-B^2. \tag{2.1e}$$

From (1.10), we have

$$A=-2(\gamma_{rr}+\gamma_{zz})+(2/r)\gamma_r, \tag{2.2a}$$

$$B=(2/r)\gamma_z, \tag{2.2b}$$

$$C=-2(\gamma_{rr}+\gamma_{zz})-(2/r)\gamma_r, \tag{2.2c}$$

$$J=4(\gamma_{rr}+\gamma_{zz})^2-(4/r^2)(\gamma_r^2+\gamma_z^2). \tag{2.2d}$$

Now equations (2.1) suggest that the elements of the matrix $\left(\begin{smallmatrix}A & B\\ B & C\end{smallmatrix}\right)$ should be taken as the components of a second-rank symmetric covariant tensor field defined on a two-dimensional manifold in the (r, z) co-ordinate basis. Form the invariant,

$$\mathrm{d}l^2=A\,\mathrm{d}r^2+2B\,\mathrm{d}r\,\mathrm{d}z+C\,\mathrm{d}z^2, \tag{2.3}$$

which shall be interpreted as a metric on the two-dimensional manifold. Choosing u and ψ as new co-ordinates (valid if and only if $D\neq 0$), the metric becomes

$$\mathrm{d}l^2=4\,\mathrm{d}u^2+\mathrm{e}^{-4u}\,\mathrm{d}\psi^2. \tag{2.4}$$

This simple metric is in fact the metric of a space of constant negative curvature, $K=-1$. When the relation, $K=-1$, is written out explicitly for the metric (2.3), we obtain equation (1.1). The left hand side of (1.1) is precisely $-4J^2(K+1)$. According to

(2.2), equation (1.1) is a fourth-order partial differential equation for γ which, when $J = 4D^2 \neq 0$, is the necessary and sufficient condition for the existence of functions, u and ψ, which satisfy all of the equations (1.9) and (1.10).

At first glance, our γ equation appears to be somewhat more complicated than the original pair of second-order equations (1.9a, b). However, this latter pair may, at best, be reduced to a single fourth-order equation for one unknown function. For example, if ψ is eliminated from (1.9a, b), there results a pair of compatible fifth-order equations for u. If an arbitrary combination of u and ψ is chosen as field variable, it is, in general, possible only to write down one fifth-order and one sixth-order equation for it. The only essentially distinct combinations of u and ψ which satisfy fourth-order field equations are $\mathscr{E} = \mathrm{e}^{2u} + \mathrm{i}\psi$ and $\mathscr{E}^* = \mathrm{e}^{2u} - \mathrm{i}\psi$. By duality, the only essentially distinct combinations of u and ω which satisfy fourth-order field equations are $r\,\mathrm{e}^{-2u} + \omega$ and $r\,\mathrm{e}^{-2u} - \omega$. Thus the γ equation is of optimum order but a strong argument in favour of the γ equation is that $\mathrm{e}^{2\gamma}$ is a much simpler function that $\mathscr{E}$ for many known solutions, especially the asymptotically flat (generalised) Tomimatsu–Sato and rotating Curzon solutions.

It is easy to transform the γ equation to arbitrary curvilinear co-ordinates. Let

$$\begin{aligned} &\rho = \rho(r, z), \qquad &\tau = \tau(r, z), \\ &r = r(\rho, \tau), \qquad &z = z(\rho, \tau) \end{aligned} \tag{2.5}$$

and define

$$A_{[\rho,\tau]} = 4u_\rho^2 + \mathrm{e}^{-4u}\psi_\rho^2, \tag{2.6a}$$

$$B_{[\rho,\tau]} = 4u_\rho u_\tau + \mathrm{e}^{-4u}\psi_\rho\psi_\tau, \tag{2.6b}$$

$$C_{[\rho,\tau]} = 4u_\tau^2 + \mathrm{e}^{-4u}\psi_\tau^2, \tag{2.6c}$$

$$D_{[\rho,\tau]} = \mathrm{e}^{-2u}(u_\rho\psi_\tau - u_\tau\psi_\rho), \tag{2.6d}$$

$$J_{[\rho,\tau]} = 4D^2 = AC - B^2. \tag{2.6e}$$

The square bracket subscript indicates the co-ordinate basis with respect to which these quantities are calculated. We shall not insist that functions or sets of functions with this notation transform as tensors under change of co-ordinate basis (e.g., D and J are relative invariants) or even transform linearly. The subscripts will be omitted whenever the co-ordinate basis is clearly understood. In the (ρ, τ) system, the property, $K = -1$, for the metric (2.4) reads

$$\begin{aligned} J(2A_{\tau\tau} + 2C_{\rho\rho} - 4B_{\rho\tau}) - J_\tau A_\tau - J_\rho C_\rho - 4J^2 - BA_\rho C_\tau + BA_\tau C_\rho + 2CA_\rho B_\tau \\ + 2AC_\tau B_\rho - 4BB_\rho B_\tau = 0. \end{aligned} \tag{2.7}$$

The tensor symmetry inherent in equations (2.1) and (2.6) does not hold for equations (2.2). So to express $A_{[\rho,\tau]}$, etc, in terms of γ, we must resort to the chain rule for partial differentiation. Writing

$$\nabla_\xi^2 = \partial^2/\partial r^2 + \partial^2/\partial z^2 + (1/r)\partial/\partial r, \tag{2.8}$$

we have

$$A_{[\rho,\tau]} = -2(r_\rho^2 + z_\rho^2)\nabla_\xi^2\gamma + 4(r_\rho/r)\gamma_\rho, \tag{2.9a}$$

$$B_{[\rho,\tau]} = -2(r_\rho r_\tau + z_\rho z_\tau)\nabla_\xi^2\gamma + 2(r_\rho/r)\gamma_\tau + 2(r_\tau/r)\gamma_\rho, \tag{2.9b}$$

$$C_{[\rho,\tau]} = -2(r_\tau^2 + z_\tau^2)\nabla_\xi^2\gamma + 4(r_\tau/r)\gamma_\tau, \tag{2.9c}$$

$$J_{[\rho,\tau]} = 4[(r_\rho z_\tau - r_\tau z_\rho)\nabla_\xi^2\gamma + (1/r)(z_\rho\gamma_\tau - z_\tau\gamma_\rho)]^2 - (4/r^2)[(z_\rho\gamma_\tau - z_\tau\gamma_\rho)^2 + (r_\rho\gamma_\tau - r_\tau\gamma_\rho)^2]. \tag{2.9d}$$

Formulae (2.9a, b, c) are written ouf explicitly in the appendix of II for several frequently used special co-ordinate systems. Note that, for an orthogonal system, $r_\rho r_\tau + z_\rho z_\tau = 0$.

Instead of taking γ as the basic field variable, we could choose A, B and C in some suitable co-ordinate basis as field variables and regard u, ψ and γ as derived quantities. One of the three field equations is, of course, (2.7). The other two arise from elimination of γ from (2.9a, b, c). In the (r, z) basis, these equations are

$$A_z - C_z = 2B_r + (2/r)B, \qquad A_r - C_r = -2B_z - (2/r)A, \tag{2.10}$$

where $A = A_{[r,z]}$, etc.

The set of Einstein's equations, (1.3a, b) and (1.4a, b, c), dual to the set, (1.9a, b) and (1.10a, b, c), relate u, ω and $\zeta \equiv \gamma - u$ and provide a fourth-order equation for ζ. Define

$$A' = 4u_r^2 - \frac{4}{r}u_r + \frac{1}{r^2} - \frac{1}{r^2}e^{4u}\omega_r^2 = -2\left(\zeta_{rr} + \zeta_{zz} - \frac{1}{r}\zeta_r\right) + \frac{1}{r^2}, \tag{2.11a}$$

$$B' = 4u_r u_z - \frac{2}{r}u_z - \frac{1}{r^2}e^{4u}\omega_r\omega_z = \frac{2}{r}\zeta_z, \tag{2.11b}$$

$$C' = 4u_z^2 - \frac{1}{r^2}e^{4u}\omega_z^2 = -2\left(\zeta_{rr} + \zeta_{zz} + \frac{1}{r}\zeta_r\right), \tag{2.11c}$$

$$D' = \frac{1}{r}\mathrm{i}\, e^{2u}\left(u_r\omega_z - u_z\omega_r - \frac{1}{2r}\omega_z\right), \tag{2.11d}$$

$$J' = 4D'^2 = A'C' - B'^2 = 4\left(\zeta_{rr} + \zeta_{zz} - \frac{1}{4r^2}\right)^2 - \frac{4}{r^2}\left(\zeta_r + \frac{1}{4r}\right)^2 - \frac{4}{r^2}\zeta_z^2. \tag{2.11e}$$

The ζ equation is precisely (1.1) with primes attached to A, B, C and J. Clearly ζ and $\gamma - \frac{1}{4}\ln r$ satisfy identical field equations.

Let us close this section by considering the cases where $J = 0$, i.e. where u and ψ are functionally related. Suppose, first, that A, B and C are not all zero. Write

$$A = 4MU_r^2, \qquad B = 4MU_rU_z, \qquad C = 4MU_z^2, \tag{2.12}$$

where $M = M(r, z)$, $U = U(r, z)$. Observe that (1.1) is identically satisfied when $J = 0$. Equations (2.10) become

$$MU_z\nabla_\xi^2U = -M_rU_rU_z + \tfrac{1}{2}M_z(U_r^2 - U_z^2), \tag{2.13a}$$

$$MU_r\nabla_\xi^2U = -\tfrac{1}{2}M_r(U_r^2 - U_z^2) - M_zU_rU_z, \tag{2.13b}$$

where ∇_ξ^2 is the operator (2.8). Eliminating ∇_ξ^2U, we obtain

$$(U_r^2 + U_z^2)(M_rU_z - M_zU_r) = 0.$$

There are two cases. If $U_r^2 + U_z^2 = 0$, then either $U = U(\alpha)$ or $U = U(\beta)$ where $\alpha = r + iz$, $\beta = r - iz$. Take $U = U(\alpha)$. Solving (2.13a, b) for M, we find $M = (\alpha + \beta)^{-1}f(\alpha)$ where f is an arbitrary function but, without loss of generality, take $f(\alpha) \equiv 1$. But now

$$A_{[\alpha,\beta]} = 4(\alpha + \beta)^{-1}[(U'(\alpha))]^2, \qquad B_{[\alpha,\beta]} = 0, \qquad C_{[\alpha,\beta]} = 0,$$

whence $\gamma = \gamma(\alpha) = \int [U'(\alpha)]^2 \,d\alpha$. Attempting to solve these equations for u and ψ, we find they are only compatible if $U'(\alpha) = 0$, i.e. if $A = B = C = 0$. Thus all solutions of (1.1) of the form, $\gamma = \gamma(\alpha)$ or $\gamma = \gamma(\beta)$, γ not a constant, are spurious.

If $M_r M_z - M_z U_r = 0$, then M and U are functionally related and so there is no loss of generality in putting $M \equiv 1$. Now

$$A = 4U_r^2, \qquad B = 4U_r U_z, \qquad C = 4U_z^2 \qquad \text{and} \qquad \nabla_\xi^2 U = 0, \tag{2.14}$$

These equations are easily solved for u and ψ, the results being

$$u = f(U), \qquad \psi = g(U),$$

where f is an arbitrary function and g is related to f by

$$f'^2 + \tfrac{1}{4} e^{-4f} g'^2 = 1. \tag{2.15}$$

Thus equation (1.1) and the set (1.10*a*, *b*, *c*) permit ψ to be an arbitrary function of u. When we apply equations (1.9*a*, *b*), we obtain additional relations between f and g,

$$f'' + \tfrac{1}{2} e^{-4f} g'^2 = 0, \qquad g'' - 4f'g' = 0,$$

for which (2.15) is a first integral. The final form of the solution is

$$e^{-2u} = \alpha_1^2 e^{-2U} + \alpha_2^2 e^{2U}, \qquad \psi e^{-2u} = -\alpha_1\alpha_3 e^{-2U} - \alpha_2\alpha_4 e^{2U}, \tag{2.16}$$

where $\alpha_1\alpha_4 - \alpha_2\alpha_3 = 1$, and is precisely the general PE solution written in a form which shows that it results from applying the transformation group **P**, given by (1.11), to the Weyl static solution with $u = U$, $\psi = 0$.

Finally, consider the case, $\gamma =$ constant, so that $A = B = C = D = 0$. Solving for u and ψ, we find

$$\text{either} \qquad e^{2u} + i\psi = \text{constant} \qquad \text{or} \qquad e^{2u} - i\psi = \text{constant}. \tag{2.17a}$$

Thus non-trivial solutions in this class must have at least one complex metric coefficient and so are not physically meaningful but they are certainly of considerable value in understanding the transformation properties of stationary axisymmetric solutions. In fact, in appendix 2, we show how one of these solutions may be mapped onto the Kerr solution. A real positive-definite Riemannian metric may always be derived by formally setting $\phi = i\phi'$, $e^{2\gamma} = -1$. Now both equations (1.9*a*, *b*) are satisfied if

$$\nabla_\xi^2(e^{-2u}) = 0. \tag{2.17b}$$

These solutions have u and ψ functionally related and depend on an arbitrary harmonic function but cannot be generated from Weyl solutions by a transformation of the form (1.11). We shall call solutions characterised by (2.17*a*, *b*) and $e^{2\gamma}$ being constant, 'PE *solutions of the second kind*', to distinguish them from the solutions, (2.16), which we shall call 'PE *solutions of the first kind*'. A number of these solutions appear on the list of solutions with non-trivial second-rank Killing tensors (see § 4 of II).

Similarly, by duality, we may distinguish Lewis solutions of the first and second kinds. The former are generated from Weyl solutions by the transformation (1.5). The latter are characterised by

$$e^{2\gamma - 2u} = kr^{-1/2}, \qquad r e^{-2u} \pm \omega = c, \qquad \nabla_\xi^2(r^{-1} e^{2u}) = 0, \tag{2.18}$$

k, c constants.

3. Construction of the full metric; uniqueness theorems

We shall give three methods for the construction of u and ψ from a given γ satisfying (1.1). This problem does not, in general, reduce to simple quadratures, as does the inverse problem, but depends on solving two independent ordinary differential equations (at best, second-order linear). Herein lies the key to the power of the γ equation to yield new solutions as a relatively simple functional form for $e^{2\gamma}$ may give rise to relatively complicated Ernst potentials.

In addition, we shall prove the following uniqueness theorems:

Theorem 1. For a given non-constant $e^{2\gamma}$ satisfying (1.1), the full set of solutions of Ernst's equations with this $e^{2\gamma}$ is generated from a particular solution by the $SL(2)$ transformation group **P** together with the possibility of changing the sign of ψ;

Theorem 2. For a given $e^{2\gamma-2u}$ satisfying the dual of (1.1), $e^{2\gamma-2u} \neq kr^{-1/2}$, k constant, the full set of solutions of Einstein's equations with this $e^{2\gamma-2u}$ is generated from a particular solution by the $SL(2)$ transformation group **L** together with the possibility of changing the sign of ω.

Theorem 2 is obviously the dual of theorem 1. The restrictions, $e^{2\gamma} \neq$ constant and $e^{2\gamma-2u} \neq kr^{-1/2}$, rule out the PE and Lewis solutions of the second kind, respectively. However, the following interesting collaries hold:

Corollary 1. If $e^{2\gamma-2u}$ satisfies the dual of (1.1) and $\nabla_\xi^2(e^{2\gamma-2u}) = 0$, then a particular solution is the PE solution of the second kind given by

$$e^{2u} = k\, e^{-2\gamma+2u}, \qquad e^{2\gamma} = k, \qquad \psi = ik\, e^{-2\gamma+2u},$$

k constant, and the general solution is generated by **L** and $\omega' = \pm\omega$;

Corollary 2. If $e^{2\gamma}$ satisfies (1.1) and $\nabla_\xi^2(r^{-1/2}\, e^{2\gamma}) = 0$, then a particular solution is the Lewis solution of the second kind given by

$$e^{2\gamma-2u} = kr^{-1/2}, \qquad e^{2u} = k^{-1} r^{1/2}\, e^{2\gamma}, \qquad \omega = -kr^{1/2}\, e^{-2\gamma},$$

k constant, and the general solution is generated by **P** and $\psi' = \pm\psi$.

We shall work in arbitrary curvilinear co-ordinates, (ρ, τ), and assume that $A = A_{[\rho,\tau]}$, etc., have been computed from equations (2.9). The ambiguity in sign of D is equivalent to the trivial ambiguity in sign of ψ and ω. Consider only $D \neq 0$ as the cases where $D = 0$ have been shown in § 2 to give rise to the PE solutions and, clearly, theorem 1 holds for the PE solutions of the first kind.

Now, from (2.6),

$$\psi_\rho = D^{-1}\, e^{2u}(Bu_\rho - Au_\tau), \qquad \psi_\tau = D^{-1}\, e^{2u}(Cu_\rho - Bu_\tau). \tag{3.1}$$

From (3.1), we may deduce linear partial differential equations for $\mathscr{E} = e^{2u} + i\psi$ and $\mathscr{E}^* = e^{2u} - i\psi$. (Note that we are allowing complex-valued metrics so that $\mathscr{E}$ and $\mathscr{E}^*$ are not necessarily complex conjugates; in fact ϵ and ϵ^* are both real for the dual of a real-valued metric.) The linear equations are

$$(B+2\mathrm{i}D)\mathscr{E}_\rho - A\mathscr{E}_\tau = 0 \qquad \text{or, equivalently,} \qquad C\mathscr{E}_\rho - (B-2\mathrm{i}D)\mathscr{E}_\tau = 0, \tag{3.2a}$$

$$(B-2\mathrm{i}D)\mathscr{E}^*_\rho - A\mathscr{E}^*_\tau = 0 \qquad \text{or, equivalently,} \qquad C\mathscr{E}^*_\rho - (B+2\mathrm{i}D)\mathscr{E}^*_\tau = 0. \tag{3.2b}$$

To solve (3.2a) and (3.2b), solve the ordinary differential equations (DE) of the first order,

$$\mathrm{d}\tau/\mathrm{d}\rho = -C^{-1}(B-2\mathrm{i}D), \qquad \mathrm{d}\tau/\mathrm{d}\rho = -C^{-1}(B+2\mathrm{i}D), \tag{3.3a, b}$$

and write their general solutions in the form,

$$\zeta(\rho, \tau) = \text{constant}, \qquad \zeta^*(\rho, \tau) = \text{constant},$$

respectively. Then the general integrals of (3.2a) and (3.2b) are, respectively,

$$\mathscr{E} = f(\zeta), \qquad \mathscr{E}^* = g(\zeta^*), \tag{3.4a, b}$$

where f and g are arbitrary functions of one variable.

The solutions (3.4a, b) are too general. Substituting into (2.6a, b, c, d), we find

$$f'g'/(f+g)^2 = \tfrac{1}{2}\,\mathrm{e}^{\Psi}, \tag{3.5a}$$

where

$$\mathrm{e}^{\Psi} = (2\zeta_\rho\zeta^*_\rho)^{-1}A = (2\zeta_\tau\zeta^*_\tau)^{-1}C. \tag{3.5b}$$

When e^{Ψ} is expressed in terms of ζ and ζ^*, it is easy to construct $f(\zeta)$ and $g(\zeta^*)$ satisfying (3.5a). The necessary and sufficient condition for the existence of $f(\zeta)$ and $g(\zeta^*)$ is that Ψ satisfy Liouville's equation,

$$\partial^2\Psi/\partial\zeta\,\partial\zeta^* = \mathrm{e}^{\Psi}. \tag{3.6}$$

(This interesting equation has general integral (3.5a), taking f and g arbitrary—see Goursat (1896, 1898), Vessiot (1942) and references cited therein for further discussion.) Equation (3.6) is equivalent to the γ equation (2.7) and, in fact, $-\mathrm{e}^{-\Psi}\Psi_{\zeta\zeta^*}$ is the Gaussian curvature of the 2-space (2.4).

To prove theorem 1, let u_0 and ψ_0 be particular solutions corresponding to a given $\mathrm{e}^{2\gamma}$. This solution may be expressed in terms of particular functions, $f_0(\zeta)$ and $g_0(\zeta)$, by

$$\mathrm{e}^{2u_0} + \mathrm{i}\psi_0 = f_0(\zeta), \qquad \mathrm{e}^{2u_0} - \mathrm{i}\psi_0 = g_0(\zeta^*).$$

To find the general solution, ignoring the sign ambiguity of D, we must find all f and g satisfying

$$f'g'/(f+g)^2 = f_0'g_0'/(f_0+g_0)^2.$$

The result is easily found to be

$$f = \frac{af_0+b}{cf_0+d}, \qquad g = \frac{ag_0-b}{-cg_0+d},$$

a, b, c, d constants, $ad-bc=1$. This transformation group is identical to (1.11) with the identification, $\alpha_1 = d$, $\alpha_2 = -\mathrm{i}c$, $\alpha_3 = \mathrm{i}b$, $\alpha_4 = a$.

The above method depends on solving the ordinary DE, (3.3a) and (3.3b). These DE will, in many cases, involve transcendental functions of both ρ and τ and so will not be readily integrated. Even in cases where γ is a function of one variable only, say $\gamma = \gamma(\tau)$, the DE will frequently take unfamiliar, complicated forms. We shall now give alternative methods for constructing the Ernst potentials which involve linear ordinary DE of the second or third order. We are free to choose the independent variable, so that if $\gamma = \gamma(\tau)$ or if $\gamma(\rho, \tau)$ depends in a simple manner on ρ, then it is highly advantageous to construct DE in which ρ is the independent variable and τ is a constant parameter.

Let us now construct a linear ordinary DE of the third order which has

$$F_1 = \mathrm{e}^{-2u}, \qquad F_2 = -\psi\, \mathrm{e}^{-2u}, \qquad F_3 = \psi^2\, \mathrm{e}^{-2u} + \mathrm{e}^{2u} \tag{3.7}$$

as linearly independent solutions. Substituting (3.1a, b) into any of (2.6a, b, c, d) and solving for u_τ, we find

$$u_\tau = BA^{-1}u_\rho + DA^{-1}(A - 4u_\rho^2)^{1/2}. \tag{3.8a}$$

Now (3.1a, b) become

$$\psi_\rho = -\mathrm{e}^{2u}(A - 4u_\rho^2)^{1/2}, \tag{3.8b}$$

$$\psi_\tau = A^{-1}\, \mathrm{e}^{2u}[4Du_\rho - B(A - 4u_\rho^2)^{1/2}]. \tag{3.8c}$$

From (3.8a), $u_{\rho\tau}$ may be expressed in terms of u_ρ and $u_{\rho\rho}$, and then, from (3.8b, c), the relation $\psi_{\rho\tau} = \psi_{\tau\rho}$ becomes the following DE for u,

$$2Au_{\rho\rho} - 4Au_\rho^2 - A_\rho u_\rho + A^2 + \Phi(A - 4u_\rho^2)^{1/2} = 0, \tag{3.9}$$

where

$$\Phi = \Phi_{[\rho,\tau]} = \tfrac{1}{4}D^{-1}(-2AB_\rho + BA_\rho + AA_\tau). \tag{3.10}$$

Thus $F = F_1$ is a particular solution of the Appell equation (Appell 1889, Cosgrove 1977c, appendix 1 below):

$$(AF_{\rho\rho} - \tfrac{1}{2}A_\rho F_\rho - A^2F)^2 + \Phi^2(F_\rho^2 - AF^2) = 0. \tag{3.11}$$

From theorem 1, the general solution of (3.11) is

$$F = \alpha_1^2 F_1 + 2\alpha_1\alpha_2 F_2 + \alpha_2^2 F_3 \tag{3.12}$$

but note that (3.11) alone without boundary conditions permits α_1 and α_2 to be arbitrary functions of τ rather than constants.

Now, Appell equations may always be converted to third-order linear equations (Appell 1889) or to Riccati or second-order linear equations (see appendix 1). Divide (3.11) throughout by $A\Phi^2$ and differentiate. The resulting expression factorises into two linear factors. The second-order factor gives rise to two spurious singular integrals and the third-order factor gives the differential equation,

$$F_{\rho\rho\rho} - \frac{\Phi_\rho}{\Phi}F_{\rho\rho} + \left[\frac{1}{4}\frac{A_\rho^2}{A^2} - \frac{1}{2}\frac{A_{\rho\rho}}{A} - A + \frac{1}{2}\frac{A_\rho\Phi_\rho}{A\Phi} + \frac{\Phi^2}{A^2}\right]F_\rho + \left[-\frac{3}{2}A_\rho + A\frac{\Phi_\rho}{\Phi}\right]F = 0, \tag{3.13}$$

which has F_1, F_2 and F_3 as linearly independent solutions.

To construct a Riccati equation from (3.9), set

$$\begin{aligned} u_\rho &= -A^{1/2}M(1 + M^2)^{-1}, \\ (A - 4u_\rho^2)^{1/2} &= A^{1/2}(1 - M^2)(1 + M^2)^{-1}. \end{aligned} \tag{3.14}$$

The function M satisfies the Riccati equation,

$$M_\rho = (\tfrac{1}{2}A^{1/2} + \tfrac{1}{2}\Phi A^{-1}) + (-\tfrac{1}{2}A^{1/2} + \tfrac{1}{2}\Phi A^{-1})M^2. \tag{3.15}$$

Next, changing variable to $L^{(\epsilon)}$, where $\epsilon = \pm 1$ is a two-valued parameter, according to

$$\frac{L_\rho^{(\epsilon)}}{L^{(\epsilon)}} = \frac{1}{2}\epsilon \mathrm{i} A^{1/2}\frac{1-\epsilon \mathrm{i} M}{1+\epsilon \mathrm{i} M}, \tag{3.16}$$

we obtain the second-order linear equation,

$$L_{\rho\rho}^{(\epsilon)} + A^{-1}(-\tfrac{1}{2}A_\rho + \epsilon \mathrm{i}\Phi)L_\rho^{(\epsilon)} + \tfrac{1}{2}\epsilon \mathrm{i} A L^{(\epsilon)} = 0. \tag{3.17}$$

The problem of finding appropriate boundary conditions for the DE, (3.13) and (3.17), will now be tackled. This leads to another DE of second-order linear or Riccati type but it is very likely that, even for very complicated solutions, this equation may yield to a simple quadrature. Let the boundary conditions be specified at $\rho = \rho_0$, ρ_0 constant (normally a curve in the (ρ, τ)-plane but may be a singular curve, a singular point or at infinity, in which case suitable asymptotic techniques must be employed).

First, we construct a Riccati equation for M in independent variable τ containing ρ as a constant parameter. From (3.8*a*), (3.9) and (3.14), we find

$$M_\tau = \bar{X} + \bar{Y}M + \bar{Z}M^2 \tag{3.18}$$

where

$$\bar{X} = \tfrac{1}{2}BA^{-1/2} + (8AD)^{-1}(BA_\tau - AC_\rho), \qquad \bar{Y} = 2DA^{-1/2},$$
$$\bar{Z} = -\tfrac{1}{2}BA^{-1/2} + (8AD)^{-1}(BA_\tau - AC_\rho).$$

Similarly, write (3.15) as

$$M_\rho = X + ZM^2 \tag{3.19}$$

where

$$X = \tfrac{1}{2}A^{1/2} + \tfrac{1}{2}\Phi A^{-1}, \qquad Z = -\tfrac{1}{2}A^{1/2} + \tfrac{1}{2}\Phi A^{-1}.$$

The compatibility conditions for (3.18) and (3.19) are

$$X_\tau - \bar{X}_\rho - X\bar{Y} = 0, \qquad -\bar{Y}_\rho + 2Z\bar{X} - 2X\bar{Z} = 0, \qquad Z_\tau - \bar{Z}_\rho + Z\bar{Y} = 0.$$

The second of these is satisfied identically. The first is identical to the third and their left hand sides are precisely $1/64D^3$ times the left hand side of the γ equation (2.7), thus providing another proof of (2.7).

Now let $M = M_0(\rho, \tau)$ be the particular solution of (3.18) and (3.19) satisfying the boundary condition,

$$M_0(\rho_0, \tau) = \mu_0(\tau) \tag{3.20}$$

where $\mu = \mu_0(\tau)$ is any particular solution of the Riccati equation,

$$\mathrm{d}\mu/\mathrm{d}\tau = \bar{X}(\rho_0, \tau) + \bar{Y}(\rho_0, \tau)\mu + \bar{Z}(\rho_0, \tau)\mu^2. \tag{3.21}$$

This Riccati equation may be converted to a second-order linear equation by standard methods but may already be linear ($\bar{Z}(\rho_0, \tau) = 0$) or of Bernoulli type ($\bar{X}(\rho_0, \tau) = 0$). Choose ρ_0 in order to achieve the simplest DE for μ and then choose the simplest solution. If $\bar{X}(\rho_0, \tau) \equiv 0$, take $\mu_0(\tau) \equiv 0$. If space–time is symmetric about the equatorial plane and if $\rho = \rho_0$ is the equatorial plane and the vectors $\partial/\partial\rho$ and $\partial/\partial\tau$ are orthogonal there (a very common situation), then $\bar{X}(\rho_0, \tau) \equiv 0$ and $\bar{Z}(\rho_0, \tau) \equiv 0$ and so $\mu_0(\tau) \equiv 0$. The general solution of (3.21) is

$$\mu = \mu_0 + \mu_1/(\mu_2 + c)$$

where c is the constant of integration and

$$\mu_1 = \exp\left\{\int_a^\tau [\bar{Y}(\rho_0, \tau') + 2\bar{Z}(\rho_0, \tau')\mu_0(\tau')]\, \mathrm{d}\tau'\right\}, \tag{3.22a}$$

$$\mu_2 = -\int_a^\tau \bar{Z}(\rho_0, \tau')\mu_1(\tau')\, \mathrm{d}\tau', \tag{3.22b}$$

where a is a fixed constant.

The general solution of (3.18) and (3.19) may be constructed by quadratures from the particular solution, $M = M_0$. Define

$$V_0 = \mu_1 \exp\left\{\int_{\rho_0}^\rho 2Z(\rho', \tau)M_0(\rho', \tau)\, \mathrm{d}\rho'\right\}, \tag{3.23a}$$

$$W_0 = -\int_{\rho_0}^\rho Z(\rho', \tau)V_0(\rho', \tau)\, \mathrm{d}\rho' + \mu_2. \tag{3.23b}$$

The functions, M_0, V_0 and W_0, satisfy the set of six useful relations:

$$M_{0\rho} = X + ZM_0^2, \qquad V_{0\rho} = 2ZM_0V_0, \qquad W_{0\rho} = -ZV_0, \tag{3.24a}$$

$$M_{0\tau} = \bar{X} + \bar{Y}M_0 + \bar{Z}M_0^2, \qquad V_{0\tau} = (\bar{Y} + 2\bar{Z}M_0)V_0, \qquad W_{0\tau} = -\bar{Z}V_0. \tag{3.24b}$$

Now the general solution of both (3.18) and (3.19) is

$$M = M_0 + V_0(W_0 + \alpha_1\alpha_2^{-1})^{-1} \tag{3.25a}$$

where the labelling, $\alpha_1\alpha_2^{-1}$, of the constant of integration anticipates the relations (1.11) between the general solution and a particular solution. In a similar spirit, define

$$V = V_0(\alpha_2 W_0 + \alpha_1)^{-2}, \tag{3.25b}$$

$$W = (\alpha_4 W_0 + \alpha_3)(\alpha_2 W_0 + \alpha_1)^{-1}. \tag{3.25c}$$

These functions, M, V and W, satisfy the same set of six relations, (3.24*a*, *b*), as M_0, V_0 and W_0.

To fix a particular solution of (3.13) for e^{-2u}, set $M = M_0$ in (3.14). Now (3.14) and (3.8*a*, *b*, *c*) may be integrated to give u and ψ explicitly in terms of M_0, V_0 and W_0. The results are

$$F_1 = V_0^{-1}(1 + M_0^2), \tag{3.26a}$$

$$F_2 = M_0 + V_0^{-1}W_0(1 + M_0^2), \tag{3.26b}$$

$$F_3 = V_0 + 2M_0W_0 + V_0^{-1}W_0^2(1 + M_0^2). \tag{3.26c}$$

If M_0, V_0 and W_0 are replaced by M, V and W, respectively, then F_1, F_2 and F_3 transform according to (1.11). This provides an independent proof of theorem 1 when $D \neq 0$. Partial derivatives of F_1, F_2 and F_3 are calculated using the relations (3.24*a*, *b*). Thus, at $\rho = \rho_0$, we have the boundary conditions:

$$F_1 = \mu_1^{-1}(1 + \mu_0^2), \tag{3.27a}$$

$$F_{1\rho} = 2(A(\rho_0, \tau))^{1/2}\mu_1^{-1}\mu_0, \tag{3.27b}$$

$$(\tfrac{1}{2}A^{-1/2}F_{1\rho})_\rho = \mu_1^{-1}[X(\rho_0, \tau) - Z(\rho_0, \tau)\mu_0^2], \tag{3.27c}$$

$$F_2 = \mu_0 + \mu_1^{-1}\mu_2(1 + \mu_0^2), \tag{3.27d}$$

$$F_{2\rho}=(A(\rho_0,\tau))^{1/2}[1+2\mu_1^{-1}\mu_2\mu_0], \tag{3.27e}$$

$$(\tfrac{1}{2}A^{-1/2}F_{2\rho})_\rho=-Z(\rho_0,\tau)+\mu_1^{-1}\mu_2[X(\rho_0,\tau)-Z(\rho_0,\tau)\mu_0^2], \tag{3.27f}$$

$$F_3=\mu_1+2\mu_2\mu_0+\mu_1^{-1}\mu_2^2(1+\mu_0^2), \tag{3.27g}$$

$$F_{3\rho}=2(A(\rho_0,\tau))^{1/2}[\mu_2+\mu_1^{-1}\mu_2^2\mu_0], \tag{3.27h}$$

$$(\tfrac{1}{2}A^{-1/2}F_{3\rho})_\rho=-Z(\rho_0,\tau)(\mu_1+2\mu_2\mu_0)+\mu_1^{-1}\mu_2^2[X(\rho_0,\tau)-Z(\rho_0,\tau)\mu_0^2]. \tag{3.27i}$$

Similarly, choose a particular solution of the $L^{(\epsilon)}$ equation (3.17) by setting $M=M_0$ in (3.16) and integrating. Another linearly independent solution is evident when we replace M_0 and V_0 by M and V, respectively. The two solutions are

$$L_1^{(\epsilon)}=\mathrm{e}^{-\epsilon\mathrm{i}Q}V_0^{-1/2}[1+\epsilon\mathrm{i}M_0], \tag{3.28a}$$

$$L_2^{(\epsilon)}=\mathrm{e}^{-\epsilon\mathrm{i}Q}V_0^{-1/2}[W_0+\epsilon\mathrm{i}(V_0+M_0W_0)], \tag{3.28b}$$

where

$$Q=\tfrac{1}{2}\int_{\rho_0}^{\rho}\Phi(\rho',\tau)(A(\rho',\tau))^{-1}\,\mathrm{d}\rho'.$$

If we replace M_0, V_0 and W_0 by M, V and W, then

$$L_1^{(\epsilon)} \text{ becomes } \alpha_1L_1^{(\epsilon)}+\alpha_2L_2^{(\epsilon)}, \qquad L_2^{(\epsilon)} \text{ becomes } \alpha_3L_1^{(\epsilon)}+\alpha_4L_2^{(\epsilon)}.$$

The relationship between $L^{(\epsilon)}$ and $L^{(-\epsilon)}$ is expressed by

$$L_{1\rho}^{(\epsilon)}=\tfrac{1}{2}\epsilon\mathrm{i}A^{1/2}\,\mathrm{e}^{-2\epsilon\mathrm{i}Q}L_1^{(-\epsilon)}, \qquad L_{2\rho}^{(\epsilon)}=\tfrac{1}{2}\epsilon\mathrm{i}A^{1/2}\,\mathrm{e}^{-2\epsilon\mathrm{i}Q}L_2^{(-\epsilon)}. \tag{3.29}$$

From these formulae, we deduce the following boundary conditions at $\rho=\rho_0$:

$$L_1^{(\epsilon)}=\mu_1^{-1/2}[1+\epsilon\mathrm{i}\mu_0], \tag{3.30a}$$

$$L_2^{(\epsilon)}=\mu_1^{-1/2}[\mu_2+\epsilon\mathrm{i}(\mu_1+\mu_2\mu_0)], \tag{3.30b}$$

$$L_{1\rho}^{(\epsilon)}=\tfrac{1}{2}\epsilon\mathrm{i}(A(\rho_0,\tau))^{1/2}\mu_1^{-1/2}[1-\epsilon\mathrm{i}\mu_0], \tag{3.30c}$$

$$L_{2\rho}^{(\epsilon)}=\tfrac{1}{2}\epsilon\mathrm{i}(A(\rho_0,\tau))^{1/2}\mu_1^{-1/2}[\mu_2-\epsilon\mathrm{i}(\mu_1+\mu_2\mu_0)]. \tag{3.30d}$$

In terms of these two solutions, explicit formulae for e^{2u} and ψ are given by

$$F_1=L_1^{(\epsilon)}L_1^{(-\epsilon)}, \qquad F_2=\tfrac{1}{2}L_1^{(\epsilon)}L_2^{(-\epsilon)}+\tfrac{1}{2}L_1^{(-\epsilon)}L_2^{(\epsilon)}, \qquad F_3=L_2^{(\epsilon)}L_2^{(-\epsilon)}. \tag{3.31}$$

Finally, we consider the effect of a change of co-ordinate basis on M, V and W from (ρ,τ) to (ρ',τ'). Regard M as $M_{[\rho,\tau]}$ etc, and write $M'=M_{[\rho',\tau']}$ etc. Let the boundary condition on M' be determined by requiring that F_1, F_2 and F_3 be invariants under a change of basis, i.e. they do not undergo a transformation (1.11).

$$M'=(PM-2DA^{-1/2}\,\partial\tau/\partial\rho')(2DA^{-1/2}(\partial\tau/\partial\rho')M+P)^{-1}, \tag{3.32a}$$

$$V'^{-1/2}[1+\epsilon\mathrm{i}M']=A'^{-1/4}(2P)^{-1/2}(P-2\epsilon\mathrm{i}DA^{-1/2}\,\partial\tau/\partial\rho')V^{-1/2}[1+\epsilon\mathrm{i}M], \tag{3.32b}$$

$$\begin{aligned}&V'^{-1/2}[W'+\epsilon\mathrm{i}(V'+M'W')]\\&\quad=A'^{-1/4}(2P)^{-1/2}(P-2\epsilon\mathrm{i}DA^{-1/2}\,\partial\tau/\partial\rho')V^{-1/2}[W+\epsilon\mathrm{i}(V+MW)],\end{aligned} \tag{3.32c}$$

where

$$A'=A_{[\rho',\tau']}=A(\partial\rho/\partial\rho')^2+2B(\partial\rho/\partial\rho')(\partial\tau/\partial\rho')+C(\partial\tau/\partial\rho')^2,$$

$$P=A'^{1/2}+A^{1/2}\,\partial\rho/\partial\rho'+BA^{-1/2}\,\partial\tau/\partial\rho'.$$

Appendix 1. Algorithm for the reduction of Appell equations to second-order linear equations†

The differential equation of Appell (1889, pp. 401–15) is the second-order second-degree equation,

$$\psi(x, y, y', y'') \equiv a_0 y''^2 + a_2 y'^2 + a_4 y^2 + 2b_1 y'y'' + 2b_2 yy'' + 2b_3 yy' = 0, \tag{A.1}$$

$a_0 = a_0(x)$ etc, $a_0 \neq 0$, the prime denoting $\mathrm{d}/\mathrm{d}x$, subject to the requirement that the general solution take the form,

$$y = h^2 y_1(x) + hky_2(x) + k^2 y_3(x), \tag{A.2}$$

where h and k are the constants of integration. The necessary and sufficient condition for (A.2) to hold is that $\lambda = \lambda(x)$ can be found such that $\mathrm{d}\psi/\mathrm{d}x - \lambda\psi$ factorises into two linear factors. One factor is of the second order of differentiation and gives rise to a pair of singular integrals of (A.1); the other yields a third-order linear DE for y with y_1, y_2 and y_3 as linearly independent solutions.

Rewrite (A.1) in the form,

$$(y'' + 2Ay' + By)^2 - (Cy'^2 + 2Dyy' + Ey^2) = 0 \tag{A.3}$$

and use the change of variable, $y = z\exp(-\int DC^{-1}\,\mathrm{d}x)$, $C \neq 0$, to put the equation in the slightly simpler form,

$$\phi(x, z, z', z'') \equiv (z'' + 2\tilde{A}z' + \tilde{B}z)^2 - (\tilde{C}z'^2 + \tilde{E}z^2) = 0. \tag{A.4}$$

(If $C = 0$ in (A.3), then (A.2) forces $D = E = 0$ also, so that (A.3) reduces trivially to a linear equation.) If (A.4) is an Appell equation, then there exists $\mu(x)$ such that $\mathrm{d}\phi/\mathrm{d}x - \mu\phi$ contains the factor $z'' + 2\tilde{A}z' + \tilde{B}z$. The conditions for this are

$$\tilde{C}' - \mu\tilde{C} + 4\tilde{A}\tilde{C} = 0, \qquad \tilde{E} - \tilde{B}\tilde{C} = 0, \qquad \tilde{E}' - \mu\tilde{E} = 0,$$

and the solutions of these equations may be expressed in the form,

$$\begin{aligned} &\tilde{A} = -\tfrac{1}{2}M'/M, \qquad \tilde{B} = -M^2, \qquad \tilde{C} = N^2, \\ &\tilde{E} = -M^2N^2, \qquad \mu = 2M'/M + 2N'/N. \end{aligned} \tag{A.5}$$

Thus a canonical form for the Appell equation is

$$[z'' - (M'/M)z' - M^2 z]^2 = N^2[z'^2 - M^2 z^2]. \tag{A.6}$$

To obtain a third-order linear equation, divide both sides of (A.6) by M^2N^2, differentiate and then remove the common factor $z'' - (M'/M)z' - M^2 z$.

Our parametrisation seems to assume that $\tilde{B} \leqslant 0$ and $\tilde{C} > 0$ but this is only for convenience. If, for example, $\tilde{C} < 0$, then one may formally set $N = \mathrm{i}\tilde{N}$, $\tilde{N}$ real, in the formulae below but instead we shall allow M and N to be either real or pure imaginary (though not both pure imaginary) as we shall be considering below complex-valued functions of the real variable x anyway. To avoid complications, restrict the domain of x to a region where M, M^{-1}, M', M'', N, N^{-1} and N' are all continuous.

The change of dependent variable,

$$U = M^{-1}z'/z, \tag{A.7}$$

† A more detailed treatment of this algorithm is given in Cosgrove (1977c).

converts (A.6) into the first-order DE,

$$[U'+M(U^2-1)]^2=N^2(U^2-1). \tag{A.8}$$

Now there are several obvious changes of variable which will convert the right hand side of (A.8) into a perfect square so that (A.8) will factorise into a pair of Riccati equations together with trivial factors which represent the singular integrals, $U=+1$ and $U=-1$. The most natural changes of variable are

$$u_1^2=(U+\epsilon_1)/(U-\epsilon_1), \qquad u_2=U+\epsilon_2(U^2-1)^{1/2}, \tag{A.9a, b}$$

$$u_3=U^{-1}+\epsilon_3(U^{-2}-1)^{1/2}, \tag{A.9c}$$

where $\epsilon_1=\pm1$, $\epsilon_2=\pm1$, $\epsilon_3=\pm1$, independently, and lead to the following pairs of Riccati equations, one for each value of $\epsilon_4=\pm1$, $\epsilon_5=\pm1$, $\epsilon_6=\pm1$,

$$u_1'=\epsilon_1 Mu_1+\tfrac{1}{2}\epsilon_4 N(1-u_1^2), \qquad u_2'=\epsilon_5 Nu_2+\tfrac{1}{2}M(1-u_2^2), \tag{A.10a, b}$$

$$u_3'=\tfrac{1}{2}M(1-u_3^2)-\tfrac{1}{2}\epsilon_6\mathrm{i}N(1+u_3^2). \tag{A.10c}$$

The following standard substitutions,

$$u_1=2\epsilon_4 N^{-1}v_1'/v_1, \qquad u_2=2M^{-1}v_2'/v_2, \tag{A.11a, b}$$

$$u_3=2(M+\epsilon_6\mathrm{i}N)^{-1}v_3'/v_3, \tag{A.11c}$$

convert these Riccati equations into the second-order linear equations,

$$v_1''+(-N'/N-\epsilon_1 M)v_1'-\tfrac{1}{4}N^2v_1=0, \tag{A.12a}$$

$$v_2''+(-M'/M-\epsilon_5 N)v_2'-\tfrac{1}{4}M^2v_2=0, \tag{A.12b}$$

$$v_3''-[(M+\epsilon_6\mathrm{i}N)'/(M+\epsilon_6\mathrm{i}N)]v_3'-\tfrac{1}{4}(M^2+N^2)v_3=0. \tag{A.12c}$$

(There are now twelve variables labelled u_1, u_2 or u_3 and all are related to each other by fractional linear transformations of the form $u_j=(au_i+b)/(cu_i+d)$, a, b, c, d real or complex constants. The reader should be aware that any variable of the form, $\tilde{u}=(au_i+b)/(cu_i+d)$, a, b, c, d functions of x, satisfies a Riccati equation and so leads to a second-order linear equation for a dependent variable of the form, $\tilde{v}=f(x)v_i+g(x)v_i'$.)

Equation (A.7) may be integrated to express z explicitly in terms of v_1, v_2 or v_3. The results are

$$z=\exp\left(-\epsilon_1\int M\,\mathrm{d}x\right)[4N^{-2}v_1'^2-v_1^2], \tag{A.13a}$$

$$z=M^{-1}\exp\left(-\epsilon_5\int N\,\mathrm{d}x\right)v_2v_2', \tag{A.13b}$$

$$z=4(M+\epsilon_6\mathrm{i}N)^{-2}v_3'^2+v_3^2. \tag{A.13c}$$

None of the sign ambiguities indicated by $\epsilon_1,\ldots,\epsilon_6$ are manifest in these formulae for z. Clearly, the general solution of the Appell equation takes the form (A.2).

Now the Appell equation (3.11) is of the form (A.6) with $M=A^{1/2}$, $N=\mathrm{i}\Phi A^{-1}$ and $x=\rho$. The function M defined by (3.14) and satisfying the Riccati equation (3.15) may be identified with u_3 with $\epsilon_3=\epsilon_6=+1$. The function $L^{(\epsilon)}$ defined by (3.16) and satisfying the linear equation (3.17) may be identified with v_2 with $\epsilon_5=-\epsilon$. Here, $u_3=(u_2-\epsilon\mathrm{i})/(1-\epsilon\mathrm{i}u_2)$.

Appendix 2

Here, it will be demonstrated that the Kerr solution can be generated from a complex-valued PE solution of the second kind by two simple elements of the transformation groups, **L** and **P**, defined by (1.5) and (1.11), respectively. Start with the PE solution of the second kind,

$$e^{2u} = -i\psi = \frac{2\kappa^2(x^2-y^2)}{px+iqy}, \qquad \omega = \frac{qx(1-y^2)+ipy(x^2-1)}{2\kappa(x^2-y^2)}, \qquad e^{2\gamma}=1, \tag{A.14}$$

where (x, y) are prolate spheroidal co-ordinates defined by $r=\kappa(x^2-1)^{1/2}(1-y^2)^{1/2}$, $z=\kappa xy$ and κ, p, q are real constants, $\kappa>0, p^2+q^2=1$. Now, apply first the element of **L** and then the element of **P** given, respectively, by

$$\begin{pmatrix}\beta_1 & \beta_2\\ \beta_3 & \beta_4\end{pmatrix} = \begin{pmatrix}0 & i\\ i & 0\end{pmatrix}, \qquad \begin{pmatrix}\alpha_1 & \alpha_2\\ \alpha_3 & \alpha_4\end{pmatrix} = 2^{-1/2}\begin{pmatrix}1 & -i\\ -i & 1\end{pmatrix}. \tag{A.15a, b}$$

The result is the (real) Kerr solution with mass $m=\kappa p^{-1}$ and angular momentum m^2q given by Ernst's formula, $\xi = px - iqy$.

What we have just done is generate a non-trivial asymptotically flat solution from a solution of Laplace's equation, $\nabla_\xi^2(e^{-2u})=0$, ∇_ξ^2 defined by (2.8). This remarkable result should surprise most those readers already most familiar with the transformation properties of stationary axisymmetric fields. The question which immediately arises is: *what are the most general boundary conditions on Laplace's equation which will permit the generation of non-trivial asymptotically flat solutions by the above method*? The answer is a little disappointing: the two-parameter Kerr solution is the unique solution to this problem (proof in Cosgrove 1978b). (This generation of the Kerr solution has been discovered independently by Herlt (1978)—I wish to thank the referees for drawing this to my attention.)

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